# 3 Existence of CR automorphisms on three dimensional CR manifolds of non-degenerate Levi form

Let $M$ be a smooth manifold of dimension $2n+1$.

**Definition 3.1** *$(H(M), J)$ is a CR-structure (of hypersurface type) if $H$ is a 2n-dimensional sub-bundle of $TM$ and $J : H \to H$ satisfies $J^2 = -id$.*

The map $J$ extends to a complex linear map of $\mathbb{C} \otimes H$ to itself and we obtain the decomposition $\mathbb{C} \otimes H = H_{1,0} \oplus H_{0,1}$ with $H_{1,0}$ as its $i$ eigenspace and $H_{0,1}$ as its $-i$ eigenspace. A CR-structure $(H(M), J)$ is called integrable if $[Z, \tilde{Z}] \in H_{1,0}$ for any local sections $Z$ and $\tilde{Z}$ of $H_{1,0}$. A manifold with an integrable CR-structure is called a CR manifold.

Now let $M$ be a CR-manifold and $\{Z_1, \ldots, Z_n\}$ be a basis for $H_{1,0}$, near some point $x$. Let $U$ be a smooth real vector field that is transversal to $H$. Then $\{Z_1, \ldots, Z_n, \overline{Z_1}, \ldots, \overline{Z_n}\}$ is a basis for $\mathbb{C} \otimes H$ near $x$. For each $i, j = 1, \ldots, n$, let

$$[Z_i, \overline{Z_j}] = \sqrt{-1} g_{ij} U \qquad \mathrm{mod} \ \ \{Z_1, \ldots, Z_n, \overline{Z_1}, \ldots, \overline{Z_n}\}.$$

Then the matrix $(g_{ij})$ is hermitian, which we call the Levi form. $M$ is called a nondegenerate CR-manifold if the matrix $(g_{ij})$ is nonsingular.

A smooth map $f$ of $M$ into another CR manifold $\tilde{M}$ is called a CR mapping if

$(i)$ $df$ maps $H(M)$ to $H(\tilde{M})$

$(ii)$ $df \circ J = J \circ df$.

We reformulate the definition of CR structures in terms of forms as follows. Given $(H(M), J)$, we choose a real nonzero form $\theta \in H^{\perp}$ and then find $\theta^1, \cdots, \theta^n$ so that $\theta, \theta^1, \cdots, \theta^n$ span $H_{0,1}^{\perp}$ linearly. Thus we have $\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n \wedge \overline{\theta^1} \wedge \cdots \wedge \overline{\theta^n} \neq 0$. Integrability can be expressed as $d\theta, d\theta^i \equiv 0$

mod $\theta, \theta^1, \dots, \theta^n$.
Conversely, given forms $\theta, \theta^1, \cdots, \theta^n$ on $M$ where

$$\begin{aligned}&\theta \text{ is real,}\\ &\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n \wedge \overline{\theta^1} \wedge \cdots \wedge \overline{\theta^n} \neq 0,\end{aligned}$$

we can define the CR structure on $M$ by setting

$$\begin{aligned}&H = \theta^\perp,\\ &H_{0,1} = \{\theta, \theta^1, \dots, \theta^n\}^\perp.\end{aligned}$$

Tanaka-Chern-Moser theory [**T1**], [**CM**] asserts that there exists a complete system of local invariants for non-degenerate CR structures. In particular, when $n = 1$ we have the following.

**Theorem 3.2 (p140 of [Ja], [BS])** *Let $M$ be a nondegenerate CR manifold of dimension 3. Then there exists an eight-dimensional bundle $Y$ over $M$ and there is a completely determined set of 1-forms $\omega$, $\omega^1$, $\phi_1^1$, $\phi^1$, $\psi$ on $Y$, of which $\omega$, $\psi$ are real and which satisfy the following :*

$$\begin{array}{lcl} d\omega &=& i\omega^1\overline{\omega^1} + \omega(\phi_1^1 + \overline{\phi_1^1}),\\ d\omega^1 &=& \omega^1\phi_1^1 + \omega\phi^1,\\ d\phi^1 &=& \frac{1}{2}\omega^1\psi + \overline{\phi_1^1}\phi^1 + Q\overline{\omega^1}\omega,\\ d\phi_1^1 &=& i\overline{\omega^1}\phi^1 + 2i\omega^1\overline{\phi^1} + \frac{1}{2}\omega\psi,\\ d\psi &=& 2i\phi^1\overline{\phi^1} + (\phi_1^1 + \overline{\phi_1^1})\psi + (R\omega^1 + \overline{R\omega^1})\omega. \end{array} \tag{3.1}$$

*Futhermore, if $M$ is another nondegenerate CR manifold with correspong notions $\tilde{Y}$, $\tilde{\omega}$, $\tilde{\omega}^1$, $\tilde{\phi}_1^1$, $\tilde{\phi}^1$, $\tilde{\psi}$, then there exists a CR diffeomorphism $f : M \to \tilde{M}$ if only if there exists a diffeomorphism $F : Y \to \tilde{Y}$ such that*

(*i*) *the diagram commutes :*

$$\begin{array}{ccc} Y & \xrightarrow{F} & \tilde{Y} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & \tilde{M} \end{array}$$

(*ii*) *Pull back of the forms, $\tilde{\omega}$, $\tilde{\omega}^1$, $\tilde{\phi}_1^1$, $\tilde{\phi}^1$, $\tilde{\psi}$ by $F$ are exactly those forms, $\omega$, $\omega^1$, $\phi_1^1$, $\phi^1$, $\psi$, respectively.*

We choose 1-forms $\omega$ and $\omega^1$ that define the CR structure of $M^3$ and that satisfy $d\omega = i\omega^1\overline{\omega^1} + \omega\phi$. This frame $\{\omega, \omega^1, \phi\}$ determines a section $\sigma : M \to Y$ and we have

$$
\begin{array}{rcl}
d\omega & = & i\omega^1\overline{\omega^1} + \omega\phi \quad \phi = \phi_1^1 + \overline{\phi_1^1}, \\
d\omega^1 & = & \omega^1\phi_1^1 + \omega\phi^1, \\
d\phi^1 & = & \frac{1}{2}\omega^1\psi + \overline{\phi_1^1}\phi^1 + q\overline{\omega^1}\omega, \\
d\phi_1^1 & = & i\overline{\omega^1}\phi^1 + 2i\omega^1\overline{\phi^1} + \frac{1}{2}\omega\psi, \\
d\psi & = & 2i\phi^1\overline{\phi^1} + (\phi_1^1 + \overline{\phi_1^1})\psi + (r\omega^1 + \overline{r}\overline{\omega^1})\omega,
\end{array}
\tag{3.2}
$$

where $q = \sigma^*Q$, $r = \sigma^*R$.

Differentiating $d\phi^1$, $d\psi$ in (3.2), we have

$$
\begin{array}{rcl}
0 & = & d^2\phi^1 \\
 & = & (dq - q\phi_1^1 - 3q\overline{\phi_1^1} - \dfrac{1}{2}\overline{r}\omega^1)\overline{\omega^1}\omega,
\end{array}
$$

$$
\begin{array}{rcl}
0 & = & d^2\psi \\
 & = & (dr - 3r\phi_1^1 - 2r\overline{\phi_1^1} + 2i\overline{q}\phi^1)\omega^1\omega + (d\overline{r} - 3\overline{r}\overline{\phi_1^1} - 2\overline{r}\phi_1^1 + 2ir\overline{\phi^1})\overline{\omega^1}\omega.
\end{array}
$$

Thus we put

$$
dq = q_0\omega + \frac{1}{2}\overline{r}\omega^1 + \overline{q_1}\overline{\omega^1} + q\phi_1^1 + 3q\overline{\phi_1^1}, \tag{3.3}
$$

$$
dr = r_0\omega + r_1\omega^1 + \tilde{r}_1\overline{\omega^1} + 3r\phi_1^1 + 2r\overline{\phi_1^1} + 2i\overline{q}\phi^1 \tag{3.4}
$$

for some $q_0$, $q_1$, $r_0$, $r_1$ and $\tilde{r}_1$ with $\tilde{r}_1$ real.

*Remark.*

1. The function $q(x)$ on $M$ is not an invariant but a relative invariant of $M$ : A different choice of $\omega$ and $\omega^1$ gives a different function $\tilde{q}(x)$, but $q(x)$ and $\tilde{q}(x)$ are either both zero or both nonzero.

2. From (3.3), we know that $q \equiv 0$ implies $r \equiv 0$. In this case, $M^3$ is CR equivalent to the real hyperquadric $Q^3$.

Let $M$ and $\tilde{M}$ be real hypersurfaces in $\mathbb{C}^n$.
A pseudo-conformal mapping of $M$ into $\tilde{M}$ is a smooth mapping that can be extended to a biholomorphism of a neighborhood of $M$ into a neighborhood of $\tilde{M}$. A pseudo-conformal mapping is obviously a CR diffeomorphism. If a hypersurface is connected and non-degenerate at a point, then the group of all pseudo-conformal automorphisms $Aut(M)$ is a Lie group of transformations with

$$\dim Aut(M) \leq n^2 + 2n,$$

and the equality holds if and only if $M$ is the real hyperquadric(See [**T1**] and [**Ya**]).

**Definition 3.3** *A smooth real vector field $X$ on $M$ is an infinitesimal CR-automorphism if $L_X V \in H$ and $L_X JV = J(L_X V)$ for any section $V$ of $H$.*

**Proposition 3.4** *Let $X$ be a smooth vector field on a CR manifold $(M, H, J)$. Then the followings are equivalent :*

*(i)* *$X$ is an infinitesimal CR-automorphism,*

*(ii)* $L_X \bar{Z} \in H_{0,1}$ *for any section* $\bar{Z}$ *of* $H_{0,1}$,

*(iii)* $L_X \omega \in H_{0,1}^{\perp}$ *for any section* $\omega$ *of* $H_{0,1}^{\perp}$.

*Proof.*

- $(i) \Rightarrow (ii)$ $J(L_X \bar{Z}) = L_X J\bar{Z} = -iL_X \bar{Z}, \forall\ \bar{Z} \in H_{0,1}$.
- $(ii) \Rightarrow (i)$ Note that $J(V + iJV) = -i(V + iJV)$ for any section $V$ of $H$ i.e. $V + iJV \in H_{0,1}$. Then $(ii)$ implies $J(L_X(V + iJV)) = -iL_X(V + iJV)$ and we get $L_X JV = J(L_X V)$ by comparing the real part of both sides.
- $(ii) \Leftrightarrow (iii)$ is easily checked. □

Let $X$ be an infinitesimal CR-automorphism on $M^3$ with $\omega(X) = \eta$ and

$\omega^1(X) = \xi$.
From the property of Lie derivative and (3.2), we have

$$\begin{aligned} L_X\omega &= d(X\lrcorner\omega) + X\lrcorner d\omega \\ &= d\eta + X\lrcorner(i\omega^1\overline{\omega^1} + \omega\phi) \\ &= d\eta + i\xi\overline{\omega^1} - i\bar{\xi}\omega^1 + \eta\phi - \phi(X)\omega, \end{aligned}$$

$$\begin{aligned} L_X\omega^1 &= d(X\lrcorner\omega^1) + X\lrcorner d\omega^1 \\ &= d\xi + X\lrcorner(\omega^1\phi_1^1 + \omega\phi^1) \\ &= d\xi + \xi\phi_1^1 - \phi_1^1(X)\omega^1 + \eta\phi^1 - \phi^1(X)\omega. \end{aligned}$$

By Proposition 3.4 we have

$$d\eta = a\omega + i\bar{\xi}\omega^1 - i\xi\overline{\omega^1} - \eta\phi, \tag{3.5}$$
$$d\xi = b\omega + c\omega^1 - \xi\phi_1^1 - \eta\phi^1 \tag{3.6}$$

for some functions $a$, $b$ and $c$.

The exterior differentiations of (3.5) and (3.6) give respectively

$$\begin{aligned} 0 &= d^2\eta \\ &= (da - i\bar{b}\omega^1 + ib\overline{\omega^1} + i\xi\overline{\phi^1} - i\bar{\xi}\phi^1 + \eta\psi)\omega + i(a - c - \bar{c})\omega^1\overline{\omega^1}, \\ 0 &= d^2\xi \\ &= (db - \eta q\overline{\omega^1} - b\overline{\phi_1^1} + \bar{c}\phi^1 + \frac{1}{2}\xi\psi)\omega + (dc - ib\overline{\omega^1} + i\bar{\xi}\phi^1 + 2i\xi\overline{\phi^1} + \frac{1}{2}\eta\psi)\omega^1. \end{aligned}$$

Thus we have

$$a = c + \bar{c}, \tag{3.7}$$
$$da = f\omega + i\bar{b}\omega^1 - ib\overline{\omega^1} + i\bar{\xi}\phi^1 - i\xi\overline{\phi^1} - \eta\psi, \tag{3.8}$$
$$db = g\omega + h\omega^1 + \eta q\overline{\omega^1} + b\overline{\phi_1^1} - \bar{c}\phi^1 - \frac{1}{2}\xi\psi, \tag{3.9}$$
$$dc = h\omega + l\omega^1 + ib\overline{\omega^1} - i\bar{\xi}\phi^1 - 2i\xi\overline{\phi^1} - \frac{1}{2}\eta\psi \tag{3.10}$$

for some functions $f$, $g$, $h$ and $l$.

From (3.7), (3.8) and (3.10), we get $l = 2i\bar{b}$ and $f = h + \overline{h}$.

Differentiating (3.10) we have

$$\begin{aligned} 0 &= d^2c \\ &= \{dh - 2i(\overline{g} + \xi\overline{q})\omega^1 - i(g + \overline{\xi}q)\overline{\omega^1} - \frac{1}{2}\eta(r\omega^1 + \overline{r}\overline{\omega^1}) - h\phi - i\overline{b}\phi^1 \\ &\quad + ib\overline{\phi^1} + \frac{1}{2}a\psi\}\omega + 2i(h - \overline{h})\omega^1\overline{\omega^1}. \end{aligned}$$

This gives $h = \overline{h}$, hence $g + \overline{\xi}q = 0$, and

$$dh = k\omega + \frac{1}{2}\eta(r\omega^1 + \overline{r}\overline{\omega^1}) + h\phi + i\overline{b}\phi^1 - ib\overline{\phi^1} - \frac{1}{2}a\psi$$

for some function $k$.

Differentiating (3.9) we have

$$\begin{aligned} 0 &= d^2b \\ &= (3\overline{c}q + cq + \eta q_0 + \frac{1}{2}\xi\overline{r} + \overline{\xi}\overline{q_1})\omega\overline{\omega^1} + (k + \frac{1}{2}\xi r + \frac{1}{2}\overline{\xi}\overline{r})\omega\omega^1, \end{aligned}$$

which implies that $k = -\frac{1}{2}\xi r - \frac{1}{2}\overline{\xi}\overline{r}$.

Thus we obtain a complete system of order 3 for $\eta$ and $\xi$, which can be expressed as

$$\left\{ \begin{aligned} d\eta &= a\omega + i\overline{\xi}\omega^1 - i\xi\overline{\omega^1} - \eta\phi, \qquad a = c + \overline{c},\ \phi = \phi_1^1 + \overline{\phi_1^1} \\ d\xi &= b\omega + c\omega^1 - \xi\phi_1^1 - \eta\phi^1 \\ db &= -\overline{\xi}q\omega + h\omega^1 + \eta q\overline{\omega^1} + b\overline{\phi_1^1} - \overline{c}\phi^1 - \tfrac{1}{2}\xi\psi \\ dc &= h\omega + 2i\overline{b}\omega^1 + ib\overline{\omega^1} - i\overline{\xi}\phi^1 - 2i\xi\overline{\phi^1} - \tfrac{1}{2}\eta\psi \\ dh &= -\tfrac{1}{2}(\xi r + \overline{\xi}\overline{r})\omega + \tfrac{1}{2}\eta(r\omega^1 + \overline{r}\overline{\omega^1}) + h\phi + i\overline{b}\phi^1 - ib\overline{\phi^1} - \tfrac{1}{2}a\psi. \end{aligned} \right. \tag{3.11}$$

Now define 1-forms on the 11-dimensional manifold $\mathcal{S} := M \times \mathbb{R} \times \mathbb{C}^3 \times \mathbb{R} = \{(x, \eta, \xi, b, c, h) | x \in M\}$ by

$$\begin{aligned} \theta^1 &= d\eta - a\omega - i\overline{\xi}\omega^1 + i\xi\overline{\omega^1} + \eta\phi \\ \theta^2 &= d\xi - b\omega - c\omega^1 + \xi\phi_1^1 + \eta\phi^1 \\ \theta^3 &= db + \overline{\xi}q\omega - h\omega^1 - \eta q\overline{\omega^1} - b\overline{\phi_1^1} + \overline{c}\phi^1 + \tfrac{1}{2}\xi\psi \\ \theta^4 &= dc - h\omega - 2i\overline{b}\omega^1 - ib\overline{\omega^1} + i\overline{\xi}\phi^1 + 2i\xi\overline{\phi^1} + \tfrac{1}{2}\eta\psi \\ \theta^5 &= dh + \tfrac{1}{2}(\xi r + \overline{\xi}\overline{r})\omega - \tfrac{1}{2}\eta(r\omega^1 + \overline{r}\overline{\omega^1}) - h\phi - i\overline{b}\phi^1 + ib\overline{\phi^1} + \tfrac{1}{2}a\psi. \end{aligned} \tag{3.12}$$

We want to apply the Frobenius theorem to $\theta^1,\ Re\,\theta^i,\ Im\,\theta^i,\ \theta^5,\quad i=2,\ 3,\ 4$

From the previous calculation we have

$$
\begin{aligned}
d\theta^1 &= 0\\
d\theta^2 &= 0\\
d\theta^3 &= -(3\overline{c}q+cq+\eta q_0+\tfrac{1}{2}\xi\overline{r}+\overline{\xi}\overline{q_1})\omega\overline{\omega^1}\\
d\theta^4 &= 0
\end{aligned}
$$

mod $\quad \theta^1,\ \theta^i,\ \overline{\theta^i},\ \theta^5,\quad i=2,\ 3,\ 4.$

We have

$$
\begin{aligned}
d\theta^5 \equiv\ & -\frac{1}{2}(\eta r_0+\xi r_1+\overline{\xi}\tilde{r}_1+(3c+2\overline{c})r+2ib\overline{q})\omega\omega^1\\
& -\frac{1}{2}(\eta\overline{r_0}+\overline{\xi}\overline{r_1}+\xi\tilde{r}_1+(3\overline{c}+2c)\overline{r}-2i\overline{b}q)\omega\overline{\omega^1}
\end{aligned}
$$

mod $\quad \theta^1,\ \theta^i,\ \overline{\theta^i},\ \theta^5,\quad i=2,\ 3,\ 4.$

Thus we put

$$
\begin{aligned}
T_1 &= 3\overline{c}q+cq+\eta q_0+\frac{1}{2}\xi\overline{r}+\overline{\xi}\overline{q_1},\\
T_2 &= \eta r_0+\xi r_1+\overline{\xi}\tilde{r}_1+(3c+2\overline{c})r+2ib\overline{q}.
\end{aligned}
$$

If $q\equiv 0$, we know that $r\equiv 0$ from the remark following Theorem 3.2 and hence $T_1\equiv T_2\equiv 0$. By the Frobenius theorem, there is a foliation of $M\times\mathbb{R}\times\mathbb{C}^3\times\mathbb{R}$ by three dimensional integral manifolds, which gives the eight-parameter family of solutions. If $q\neq 0$, we solve $T_1=T_2=0$ to get $b=b(x,\eta,\xi)$, $c=c(x,\eta,\xi)$ and (3.11) is reduced to a complete system of order 1 :

$$
\left\{
\begin{aligned}
d\eta &= (c(x,\eta,\xi)+\overline{c(x,\eta,\xi)})\omega+i\overline{\xi}\omega^1-i\xi\overline{\omega^1}-\eta\phi,\\
d\xi &= b(x,\eta,\xi)\omega+c(x,\eta,\xi)\omega^1-\xi\phi_1^1-\eta\phi^1.
\end{aligned}
\right.
$$

We may keep analyzing this pfaffian system on $\mathcal{S}':=M\times\mathbb{R}\times\mathbb{C}$ by checking the Frobenius condition. If the system is integrable on $\mathcal{S}'$ in the sense of Frobenius there exits 3 parameter family of solutions. We summarize the above discussions of this section in the following

**Theorem 3.5** *Let $M$ be a nondegenerate CR manifold of dimension 3. Let $q$ be the relative CR invariant as in (3.2). Then*

(*i*) *If $q \equiv 0$, $M^3$ is CR equivalent to the real hyperquadric $Q^3$ and there exist eight-parameter family of infinitesimal CR automorphisms.*

(*ii*) *If $q \neq 0$, we obtain a complete system of order 1 for infinitesimal CR automorphisms :*

$$\begin{cases} d\eta &= (c(x,\eta,\xi) + \overline{c(x,\eta,\xi)})\omega + i\bar{\xi}\omega^1 - i\xi\overline{\omega^1} - \eta\phi, \\ d\xi &= b(x,\eta,\xi)\omega + c(x,\eta,\xi)\omega^1 - \xi\phi_1^1 - \eta\phi^1, \end{cases}$$

*where $b$ and $c$ are functions on $M \times \mathbb{R} \times \mathbb{C}$ given by $T_1 = T_2 = 0$. In particular, there are infinitesimal CR automorphisms on $M$ of at most three parameters.*

As for the dimension of CR automorphism group of $M$ we refer [**Ja**]. The second part of Theorem 3.5 can also be proved by using Theorem 2 and its corollary of [**Ja**].

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